Approximation of subharmonic functions in the unit disk

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1 Introduction

We use the standard notions of subharmonic function theory [1]. Let us introduce some notation. Let $U(E,t)=\{\zeta\in\mathbb{C}:\operatorname{dist}(\zeta,E)<t\}$, $E\subset\mathbb{C}$, $t>0$, where $\operatorname{dist}(z,E)\stackrel{\mathrm{def}}{=}\inf_{\zeta\in E}|z-\zeta|$, and $U(z,t)\equiv U(\{z\},t)$ for $z\in\mathbb{C}$. The class of subharmonic functions in a domain $G\subset\mathbb{C}$ is denoted by $\mathrm{SH}(G)$. For a subharmonic function $u\in\mathrm{SH}(U(0,R))$, $0<R\le+\infty$ we write $B(r,u)=\max\{u(z):|z|=r\}$, $0<r<R$, and define the order $\rho[u]$ by
$\rho[u]=\limsup\limits_{r\to+\infty}\log B(r,u)/\log r$ if $R=\infty$ and by $\sigma[u]=\limsup\limits_{r\to R}\log B(r,u)/\log\frac{1}{R-r}$ if $R<\infty$.

Let also μ_u denote the Riesz measure associated with the subharmonic function u, $n(r,u)=\mu_u(\overline{U(0,r)})$, m be the planar Lebesgue measure, l be the Lebesgue measure on the positive ray. For an analytic function f in $\mathbb{D}$ we write $Z_f=\{z\in\mathbb{D}:f(z)=0\}$. The symbol $C(\cdot)$ with indices stands for some positive constants depending only on values in the brackets. We write $a\asymp b$ if $C_1a\le b\le C_2a$ for some positive constants C_1 and C_2, and $a(r)\sim b(r)$ if $\lim_{r\to R}a(r)/b(r)=1$.

An important result was proved by R. S. Yulmukhametov [2]. *For any function $u\in\mathrm{SH}(\mathbb{C})$ of order $\rho\in(0,+\infty)$, and $\alpha>\rho$ there exist an entire function f and a set $E_\alpha\subset\mathbb{C}$ such that*

$$\big|u(z)-\log|f(z)|\big|\le C(\alpha)\log|z|,\quad z\to\infty,\ z\notin E_\alpha,\tag{1.1}$$

and E_α can be covered by a family of disks $U(z_j,t_j)$, $j\in\mathbb{N}$, with $\sum_{|z_j|>R}t_j=O(R^{\rho-\alpha})$, $(R\to+\infty)$.

If $u\in\mathrm{SH}(\mathbb{D})$ a counterpart of (1.1) holds with $\log\frac{1}{1-|z|}$ instead of $\log|z|$ and an apropriate choice of E_α.

From the recent result of Yu. Lyubarskii and Eu. Malinnikova [3] it follows that for L_1 approximation relative to planar measure, we may drop the assumption that u has finite order of growth, and obtain sharp estimates.

Theorem A ([3])**.** *Let $u \in \mathrm{SH}(\mathbb{C})$. Then, for each $q > 1/2$, there exist $R_0 > 0$ and an entire function f such that*

$$\frac{1}{\pi R^2} \int\limits_{|z|<R} \big|u(z) - \log|f(z)|\big| dm(z) < q \log R, \quad R > R_0. \tag{1.2}$$

An example constructed in [3] shows that we cannot take $q < 1/2$ in estimate (1.2). The case $q = 1/2$ remains open.

The following theorem complements this result.

Let Φ be the class of slowly growing functions $\psi\colon [1, +\infty) \to (1, +\infty)$ (in particular, $\psi(2r) \sim \psi(r)$ as $r \to +\infty$).

Theorem B ([4])**.** *Let $u \in \mathrm{SH}(\mathbb{C})$, $\mu = \mu_u$. If for some $\psi \in \Phi$ there exists a constant R_1 satisfying the condition*

$$(\forall R > R_1) : \mu(\{z : R < |z| \le R\psi(R)\}) > 1, \tag{1.3}$$

then there exists an entire function f such that $(R \ge R_1)$

$$\int\limits_{|z|<R} \big|u(z) - \log|f(z)|\big|\, dm(z) = O(R^2 \log \psi(R)). \tag{1.4}$$

Remark 1.1. In the case $\psi(r) \equiv q > 1$ we obtain Theorem 1 [3].

The following example and Theorem C show (see [4] for details) that estimate (1.4) is sharp in the class of subharmonic functions satisfying (1.3).

For $\varphi \in \Phi$, let

$$u(z) = u_\varphi(z) = \frac{1}{2} \sum_{k=1}^{+\infty} \log\Big|1 - \frac{z}{r_k}\Big|,$$

where $r_0 = 2$, $r_{k+1} = r_k \varphi(r_k)$, $k \in \mathbb{N} \cup \{0\}$. Thus, μ_u satisfies condition (1.3) with $\psi(x) = \varphi^3(x)$.

Theorem C. *Let $\psi \in \Phi$ be such that $\psi(r) \to +\infty$ $(r \to +\infty)$. There exists no entire function f for which*

$$\int\limits_{|z|<R} \big|u_\psi(z) - \log|f(z)|\big|\, dm(z) = o(R^2 \log \psi(R)), \quad R \to \infty.$$

A further question appears naturally: Are there counterparts of Theorems A and B for subharmonic functions in the unit disk? We have the following theorem.

Theorem 1. *Let $u \in \mathrm{SH}(\mathbb{D})$. There exist an absolute constant C and an analytic function f in $\mathbb{D}$ such that*

$$\int_{\mathbb{D}} \big|u(z) - \log|f(z)|\big|\, dm(z) < C. \tag{1.5}$$

For a measurable set $E \subset [0,1)$ we define the density

$$\mathcal{D}_1 E = \varlimsup_{R \uparrow 1} \frac{l(E \cap [R,1))}{1-R}.$$

Corollary 1. *Let $u \in \mathrm{SH}(\mathbb{D})$, $\varepsilon > 0$. There exist an analytic function f in $\mathbb{D}$ and $E \subset [0,1)$, $\mathcal{D}_1 E < \varepsilon$, such that*

$$\int_0^{2\pi} \big|u(re^{i\theta}) - \log|f(re^{i\theta})|\big| d\theta = O\Big(\frac{1}{1-r}\Big), \quad r \uparrow 1, r \notin E. \tag{1.6}$$

Relationship (1.6) is equivalent to the condition

$$T(r,u) - T(r, \log|f|) = O((1-r)^{-1}), \quad r \uparrow 1, r \notin E,$$

where $T(r,v)$ is the Nevanlinna characteristic of a subharmonic function v. The author does not know whether (1.6) is best possible.

Remark 1.2. No restriction on the Riesz measure μ_u or the growth of u is required in Theorem 1.

Remark 1.3. It is clear that (1.5) is sharp in the class $\mathrm{SH}(\mathbb{D})$, but under growth restrictions can be improved.

Theorem D (Hirnyk [5])**.** *Let $u \in \mathrm{SH}(\mathbb{D})$, $\sigma[u] < +\infty$. Then there exists an analytic function f in $\mathbb{D}$ such that*

$$\int_0^{2\pi} \big|u(re^{i\theta}) - \log|f(re^{i\theta})|\big| d\theta = O\Big(\log^2 \frac{1}{1-r}\Big), \quad r \uparrow 1.$$

Theorem 1 does not allow the conclusion that

$$u(z) - \log|f(z)| = O(1), \quad z \in \mathbb{D} \setminus E \tag{1.7}$$

for any "small" set E.

Sufficient conditions for (1.7) in the complex plane were obtained in [3]. It uses so called notion of a *locally regular measure* which admits a *partition of slow variation.*

We also prove a counterpart of Theorem 3′ of [3] using a similar concept. A corresponding Theorem 3 will be formulated in section 3. Here we formulate an application of Theorem 3.

Theorem 2. *Let $\gamma_j = (z = z_j(t) : t \in [0,1])$, $1 \le j \le m$ be smooth Jordan curves in $\overline{U(0,1)}$ such that $\arg z_j(t) = \theta_j(|z_j(t)|) \equiv \theta_j(r)$, $|z_j(1)| = 1$, $|\theta_j'(r)| \le K$ for $r_0 \le r < 1$ and some constants $r_0 \in (0,1)$, $K > 0$, $1 \le j \le m$. Let $u \in SH(\mathbb{D})$, supp $\mu_u \subset \bigcup_{j=1}^m [\gamma_j]$, $\mu_u([\gamma_j] \cap [\gamma_k]) = 0$, $j \ne k$, and*

$$\mu_u\Big|_{[\gamma_j]}(U(0,r)) = \frac{\Delta_j}{(1-r)^{\sigma(r)}},$$

where Δ_j is a positive constant, $\sigma(r) = \rho\big(\frac{1}{1-r}\big)$, $\rho(R)$ is a proximate order [7], $\rho(R) \to \sigma > 0$ as $R \to +\infty$.

Then there exists an analytic function f such that for all $\varepsilon > 0$

$$\log|f(z)| - u(z) = O(1), \tag{1.8}$$

$z \notin E_\varepsilon = \{\zeta \in \mathbb{D} : \mathrm{dist}\,(\zeta, Z_f) \le \varepsilon(1-|\zeta|)^{1+\sigma(r)}\}$, where

$$\log|f(z)| - u(z) \le C, \tag{1.9}$$

for some $C > 0$ and all $z \in \mathbb{D}$. Moreover,

$$Z_f \subset \bigcup_{\zeta \in \bigcup_j [\gamma_j]} U(\zeta, 2(1-|\zeta|)^{1+\sigma(r)}),$$

and

$$T(r,u) - T(r,f) = O(1), \quad r \uparrow 1. \tag{1.10}$$

Remark 1.4. Obviously, we can't obtain a lower estimate for the left-hand side of (1.9) for all z, because it equals $-\infty$ on Z_f.

Theorems similar to Theorem 2 are proved in [6, Ch.10,Th.10.16,10.20]. The difference is that in [6] only more crude estimates are obtained for approximation in a more general settings.

2 Proof of Theorem 1

2.1. Preliminaries

Let $u \in \mathrm{SH}(\mathbb{D})$. Then the Riesz measure μ_u is finite on compact subsets of $\mathbb{D}$. In order to apply a partition theorem (Theorem E) we have to modify the Riesz measure. On subtracting an integer-valued discrete measure $\tilde{\mu}$ from μ_u we may arrange that $\nu(\{p\}) = (\mu_u - \tilde{\mu})(\{p\}) < 1$ for any point $p \in \mathbb{D}$. The measure $\tilde{\mu}$ corresponds to the zeros of an entire function g. Thus we can consider $\tilde{u} = u - \log|g|$, $\mu_{\tilde{u}} = \nu$. According to Lemma 1 [4] in any neighbourhood of the origin there exists a point z_0 with the following properties:

a) on each line L_α going through z_0 there is at most one point ζ_α such that $\nu(\{\zeta_\alpha\}) > 0$, while $\nu(L_\alpha \setminus \{\zeta_\alpha\}) = 0$;

b) on each circle K_ρ with center z_0 there exists at most one point ζ_ρ such that $\nu(\{\zeta_\rho\}) > 0$, while $\nu(K_\rho \setminus \{\zeta_\rho\}) = 0$.

As it follows from the proof of Lemma 1 [4], the set of points z_0 that do not satisfy one of the conditions a) and b) has planar measure zero. The similar assertion holds for the polar set $u(z_0) = -\infty$ [1, Chap.5.9,Theorem 5.32]. Therefore, we can assume that properties a), b) hold, and $u(z_0) \neq -\infty$.

Then consider the subharmonic function $u_0(z) = u\left(\frac{z_0 - z}{1 - z\bar{z}_0}\right) \equiv u(w(z))$, $u_0(0) = u(z_0)$. Since $|w'(z)| = \frac{1-|z_0|^2}{|1-z\bar{z}_0|^2}$, we have $||w'(z)| - 1| \leq 3|z_0|$ for $|z_0| \leq 1/2$.

The Jacobian of the transformation $w(z)$ is $|w'(z)|^2$, consequently this change of variables doesn't change relation (1.5).

Let

$$u_3(z) = \int_{U(0,1/2)} \log|z - \zeta|\, d\mu_u(\zeta). \tag{2.1}$$

The subharmonic function $u(z) - u_3(z)$ is harmonic in $U(0, 1/2)$.

Let $q \in (0, 1)$ be such that

$$\sum_{j=1}^{12} q^j > 11. \tag{2.2}$$

We define ($n \in \{0, 1, \dots\}$)

$$R_n = 1 - q^n/2,\ A_n = \{\zeta : R_n \leq |\zeta| < R_{n+1}\},\ M_n = M_n(q) = \left[\frac{2\pi}{\log \frac{R_{n+1}}{R_n}}\right],$$

$$A_{n,m} = \left\{\zeta \in A_n : \frac{2\pi m}{M_n} \leq \arg_0 \zeta < \frac{2\pi(m+1)}{M_n}\right\}, \quad 0 \leq m \leq M_n - 1.$$

Represent $\mu_u\Big|_{A_{n,m}} = \mu_{n,m}^{(1)} + \mu_{n,m}^{(2)}$ such that

i) $\operatorname{supp} \mu_{n,m}^{(j)} \subset \overline{A}_{n,m}$, $j \in \{1, 2\}$;

ii) $\mu_{n,m}^{(1)}(\overline{A}_{n,m}) \in 2\mathbb{Z}_+$, $0 \leq \mu_{n,m}^{(2)}(\overline{A}_{n,m}) < 2$.

Let

$$\mu_n^{(j)} = \sum_{m=0}^{M_n - 1} \mu_{n,m}^{(j)}, \quad \tilde{\mu}^{(j)} = \sum_n \mu_n^{(j)},\ j \in \{1, 2\}.$$

Property ii) implies

$$\mu_n^{(2)}(\overline{A_n}) \le \frac{13}{(1-q)(1-R_n)}, \quad n \to +\infty, \tag{2.3}$$

as follows from the asymptotic equality

$$\log \frac{R_{n+1}}{R_n} \sim (1-q)(1-R_n), \quad n \to +\infty, \tag{2.4}$$

and the definition of M_n.

Let

$$u_2(z) = \int_{\mathbb{D}} \log\Big|E\Big(\frac{1-|\zeta|^2}{1-\bar\zeta z}, 1\Big)\Big| d\tilde\mu^{(2)}(\zeta), \tag{2.5}$$

where $E(w,p) = (1-w)\exp\{w + w^2/2 + \cdots + w^p/p\}$, $p \in \mathbb{N}$ is the Weierstrass primary factor.

Lemma 1. *$u_2 \in SH(\mathbb{D})$, and*

$$T(r,u_2) = O\Big(\log^2 \frac{1}{1-r}\Big),\ r \uparrow 1, \quad \int_{\mathbb{D}} |u_2(z)|\, dm(z) < C_1(q).$$

Proof of Lemma 1. The following estimates for $\log|E(w,p)|$ are well-known (cf. [7, Ch.1, §4, Lemma 2], [1, Ch.4.1, Lemma 4.2])

$$\begin{gathered} |\log E(w,1)| \le \frac{|w|^2}{2(1-|w|)}, \quad |w|<1, \\ \log|E(w,1)| \le 6e|w|^2, \quad w \in \mathbb{C}. \end{gathered} \tag{2.6}$$

First, we prove convergence of the integral in (2.5). For fixed R_n let $|z| \le R_n$. We choose p such that $q^p < 1/4$. Then for $|\zeta| \ge R_{n+p}$ we have

$$\frac{1-|\zeta|^2}{|1-\bar\zeta z|} \le \frac{2(1-|\zeta|)}{1-|z|} \le \frac{2(1-R_{n+p})}{1-R_n} < \frac{1}{2}.$$

Hence, using the first estimate (2.6), (2.3) and the definition of R_n we obtain

$$\int_{|\zeta|\ge R_{n+p}} \Big|\log\Big|E\Big(\frac{1-|\zeta|^2}{1-\bar\zeta z},1\Big)\Big|\Big|\, d\tilde\mu^{(2)}(\zeta) \le \int_{|\zeta|\ge R_{n+p}} \Big(\frac{2(1-|\zeta|)}{1-|z|}\Big)^2 d\tilde\mu^{(2)}(\zeta) \le$$
$$\le \frac{4}{(1-|z|)^2}\sum_{k=n+p}^{\infty}(1-R_k)^2\int_{\bar A_k} d\tilde\mu^{(2)}(\zeta) \le \frac{52}{(1-q)(1-|z|)^2}\sum_{k=n+p}^{\infty}(1-R_k) =$$
$$= \frac{52(1-R_{n+p})}{(1-q)^2(1-|z|)^2} \le \frac{C_2(q)}{1-R_n}.$$

Thus, u_2 is represented by the integral of the subharmonic function $\log|E|$ of z, and the integral converges uniformly on compact subsets in $\mathbb{D}$, and so $u_2 \in \mathrm{SH}(\mathbb{D})$. Since $1-|\zeta|^2 \le 3/4$ for $\zeta \in \operatorname{supp} \tilde{\mu}^{(2)}$, using (2.6) and (2.3) we have

$$|u_2(0)| \le \int_{\mathbb{D}} |\log|E(1-|\zeta|^2,1)||\, d\tilde{\mu}^{(2)}(\zeta) \le \int_{\mathbb{D}} 2(1-|\zeta|^2)^2 d\tilde{\mu}^{(2)}(\zeta) \le$$

$$\le 8\sum_{k=0}^{\infty} \int_{\bar{A}_k} (1-|\zeta|)^2 d\tilde{\mu}^{(2)}(\zeta) \le \frac{104}{1-q} \sum_{k=0}^{\infty} (1-R_k) = C_3(q). \tag{2.7}$$

Let us estimate $T(r,u_2) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} u_2^+(re^{i\theta})\, d\theta$ for $r \le R_n$, where $u^+ = \max\{u, 0\}$. Note that for $|\zeta| \le R_{n+2}$, $|z| \le R_n$ we have $\frac{1-|\zeta|^2}{|1-\bar{\zeta}z|} \le 2$. Thus

$$\log\Big|E\Big(\frac{1-|\zeta|^2}{1-\bar{\zeta}z},1\Big)\Big| \le 12e\frac{1-|\zeta|^2}{|1-\bar{\zeta}z|}$$

in this case. Using the latter estimate, (2.6), (2.3), and the lemma [10, Ch.5.10, p.226] we get

$$T(r,u_2) \le \frac{1}{2\pi}\int_0^{2\pi} \Big(\sum_{k=0}^{n+1} \int_{\bar{A}_k} 12e\frac{1-|\zeta|^2}{|1-\bar{\zeta}re^{i\theta}|}\, d\mu_k^{(2)}(\zeta) \Big) d\theta +$$

$$+\frac{1}{2\pi}\int_0^{2\pi} \Big(\sum_{k=n+2}^{\infty} \int_{\bar{A}_k} 6e\frac{(1-|\zeta|^2)^2}{|1-\bar{\zeta}re^{i\theta}|^2}\, d\mu_k^{(2)}(\zeta) \Big) d\theta \le$$

$$\le C_4(q)\Big(\sum_{k=0}^{n+1} \int_{\bar{A}_k} (1-|\zeta|^2)\log\frac{1}{1-r} d\mu_k^{(2)}(\zeta) + \sum_{k=n+2}^{\infty} \int_{\bar{A}_k} \frac{(1-|\zeta|^2)^2}{1-r} d\mu_k^{(2)}(\zeta) \Big) \le$$

$$\le C_5(q)\Big(\sum_{k=0}^{n+1} \log\frac{1}{1-r} + \sum_{k=n+2}^{\infty} \frac{1-R_k}{1-r} \Big) \le$$

$$\le C_6(q) n \log\frac{1}{1-r} \le C_7(q)\log^2\frac{1}{1-r}.$$

Finally, by the First main theorem for subharmonic functions [1, Ch. 3.9]

$$m(r,u_2) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_0^{2\pi} u_2^-(re^{i\theta}) d\theta =$$

$$= T(r,u_2) - \int_0^r \frac{n(t,u_2)}{t} dt - u_2(0) \le T(r,u_2) + C_3(q).$$

Therefore, $\int_0^{2\pi} |u_2(re^{i\theta})| d\theta \le 4\pi T(r,u_2) + C_8(q)$.

Consequently,

$$\int_{|z|\le 1} |u_2(z)|\, dm(z) \le 4\pi \int_0^1 T(r,u_2)\, dr + C_8(q) \le$$

$$\leq C_9(q)\int_0^1 \log^2\frac{1}{1-r}dr \leq C_{10}(q).$$

Lemma 1 is proved. □

2.2. Approximation of $\tilde{\mu}_1$

The following theorem plays a key role in approximation of u.

Theorem E. *Let μ be a measure in $\mathbb{R}^2$ with compact support,* $\operatorname{supp}\mu \subset \Pi$, *and $\mu(\Pi) \in \mathbb{N}$, where Π is a rectangle with ratio of side lengths $l_0 \geq 1$. Suppose, in addition, that for any line L parallel to a side of Π, there is at most one point $p \in L$ such that*

$$0 < \mu(\{p\})(< 1) \quad \text{while always} \quad \mu(L \setminus \{p\}) = 0, \tag{2.8}$$

Then there exist a system of rectangles $\Pi_k \subset \Pi$ with sides parallel to the sides of Π, and measures μ_k with the following properties:

1) $\operatorname{supp}\mu_k \subset \Pi_k$;
2) $\mu_k(\Pi_k) = 1$, $\sum_k \mu_k = \mu$;
3) *the interiors of the convex hulls of the supports of μ_k are pairwise disjoint;*
4) *the ratio of the side lengths of rectangles Π_k lies in the interval $[1/l, l]$, where $l = \max\{l_0, 3\}$;*
5) *each point of the plane belongs to the interiors of at most 4 rectangles Π_k.*

Theorem E was proved by R. S. Yulmukhametov [2, Theorem 1] for absolutely continuous measures (i.e. ν such that $m(E) = 0 \Rightarrow \nu(E) = 0$) and $l_0 = 1$. In this case condition (2.8) is fulfilled automatically. In [8, Theorem 2.1] D. Drasin showed that Yulmukhametov's proof works if the condition of continuity is replaced by condition (2.8). We can drop condition (2.8) rotating the initial square [8]. One can also consider Theorem E as a formal consequence of Theorem 3 [4]. Here l_0 plays role for a finite set of rectangles corresponding to small k's, but in [4] it plays the principal role in the proof.

Remark 2.1. In the proof of Theorem E [8] rectangles Π_k are obtained by splitting the given rectangles, starting with Π, into smaller rectangles in the following way. The length of the smaller side of the initial rectangle coincides with that of a side of the rectangle obtained in the first generation, and the length of the other side of the new rectangle is between one third and two thirds of the length of the other side of the initial rectangle. Thus we can start with a rectangle instead of a square and $l = \max\{l_0, 3\}$.

Let $u_1(z) = u(z) - u_2(z) - u_3(z)$. Then $\mu_{u_1} = \tilde{\mu}^{(1)}$, $\mu^{(1)}_{n,m}(\bar{A}_{n,m}) \in 2\mathbb{Z}_+$, $n \in \mathbb{Z}_+$, $0 \le m \le M_n - 1$.

Let

$$P_{n,m} = \log \overline{A}_{n,m} =$$
$$= \Big\{ s = \sigma + it : \log R_n \le \sigma \le \log R_{n+1}, \frac{2\pi m}{M_n} \le t \le \frac{2\pi(m+1)}{M_n} \Big\}.$$

According to (2.4) the ratio of the sides of $P_{n,m}$ is

$$\frac{\log \frac{R_{n+1}}{R_n}}{2\pi/[\frac{2\pi}{(1-q)(1-R_n)}]} \to 1, \quad n \to \infty. \tag{2.9}$$

Let $d\nu_{n,m}(s) \overset{\text{def}}{=} d\mu^{(1)}_{n,m}(e^s)$, $s \in P_{n,m}$, (i. e. $\nu_{n,m}(S) = \mu^{(1)}_{n,m}(\exp S)$ for every Borel set $S \subset \mathbb{C}$). By our assumptions the conditions of Theorem E are satisfied for $\Pi = P_{n,m}$ and $\mu = \nu_{n,m}/2$, and all admissible n, m. By Theorem E there exists a system (P_{nkm}, ν_{nmk}) of rectangles and measures, $k \le N_{nm}$, $0 \le m \le M_n - 1$ with the properties: 1) $\nu_{nmk}(P_{nmk}) = 1$; 2) $\operatorname{supp} \nu_{nmk} \subset P_{nmk}$; 3) $2\sum_k \nu_{nmk} = \nu_{n,m}$; 4) every point s such that $\operatorname{Re} s < 0$, $0 \le \operatorname{Im} s < 2\pi$ belongs to the interiors of at most four rectangles P_{nmk}; 5) the ratio of the side lengths lies between two positive constants. Indexing the new system $(P_{nmk}, 2\nu_{nmk})$ with the natural numbers, we obtain a system $(P^{(l)}, \nu^{(l)})$ with $\nu^{(l)}(P^{(l)}) = 2$, $\operatorname{supp} \nu^{(l)} \subset P^{(l)}$ etc.

Let the measure $\mu^{(l)}$ defined on $\mathbb{D}$ be such that $d\mu^{(l)}(e^s) \overset{\text{def}}{=} d\nu^{(l)}(s)$, $\operatorname{Re} s < 0$, $0 \le \operatorname{Im} s < 2\pi$, $Q^{(l)} = \exp P^{(l)}$. Let

$$\zeta_l \overset{\text{def}}{=} \frac{1}{2} \int\limits_{Q^{(l)}} \zeta \, d\mu^{(l)}(\zeta). \tag{2.10}$$

be the center of mass of $Q^{(l)}$, $l \in \mathbb{N}$.

We define $\zeta^{(1)}_l$, $\zeta^{(2)}_l$ as solutions of the system

$$\begin{cases} \zeta^{(1)}_l + \zeta^{(2)}_l = \int\limits_{Q^{(l)}} \zeta \, d\mu^{(l)}(\zeta), \\ (\zeta^{(1)}_l)^2 + (\zeta^{(2)}_l)^2 = \int\limits_{Q^{(l)}} \zeta^2 \, d\mu^{(l)}(\zeta), \end{cases} \tag{2.11}$$

From (2.11) and (2.10) it follows that (see [3], [4] or Lemma 3 below)

$$|\zeta^{(j)}_l - \zeta_l| \le \operatorname{diam} Q^{(l)} \equiv d_l, \quad j \in \{1, 2\}.$$

Consequently we obtain

$$\max_{\zeta\in Q^{(l)}} |\zeta-\zeta_l^{(j)}|\le 2d_l,\ j\in\{1,2\},\qquad \sup_{\zeta\in Q^{(l)}}|\zeta-\zeta_l|\le d_l. \tag{2.12}$$

We write

$$\Delta_l(z)\stackrel{\text{def}}{=}\int\limits_{Q^{(l)}}\Big(\log\Big|\frac{z-\zeta}{1-z\bar\zeta}\Big|-\frac12\log\Big|\frac{z-\zeta_l^{(1)}}{1-z\bar\zeta_l^{(1)}}\Big|-\frac12\log\Big|\frac{z-\zeta_l^{(2)}}{1-z\bar\zeta_l^{(2)}}\Big|\Big)\,d\mu^{(l)}(\zeta),$$

$$V(z)\stackrel{\text{def}}{=}\sum_l\Delta_l(z).$$

Fix a sufficiently large m (in particular, $m\ge 13$) and $z\in A_m$. Let $\mathcal{L}^+$ be the set of l's such that $Q^{(l)}\subset U(0,R_{m-13})$, and $\mathcal{L}^-$ the set of l's with $Q^{(l)}\subset\{\zeta: R_{m+13}\le|\zeta|<1\}$, $\mathcal{L}^0=\mathbb{N}\setminus(\mathcal{L}^-\cup\mathcal{L}^+)$.

Lemma 2. *There exists $l^*\in\mathbb{N}$ such that $\zeta_1,\ \zeta_2\in U(Q^{(l)},2d_l)$, $l\in\mathcal{L}^+\cup\mathcal{L}^-$, $l>l^*$ imply*

$$\frac{1}{16}|z-\zeta_2|\le|z-\zeta_1|\le 16|z-\zeta_2|.$$

Proof of Lemma 2. First, let $l\in\mathcal{L}^+$, i.e. $z\in A_m$, $Q^{(l)}\subset\overline{A_p}$, $p\le m-13$. In view of (2.9) $Q^{(l)}=\exp P^{(l)}$ is "almost a square". More precisely, there exists $l^*\in\mathbb{N}$ such that for all $l>l^*$

$$\operatorname{diam}Q^{(l)}=d_l<\frac32(R_{p+1}-R_p),\quad Q^{(l)}\subset\overline{A_p}.$$

Since $\zeta_1,\ \zeta_2\in U(Q^{(l)},2d_l)$, we have

$$R_p-3(R_{p+1}-R_p)\le|\zeta_2|\le R_{p+1}+3(R_{p+1}-R_p), \tag{2.13}$$

$$|z-\zeta_1|\ge|z-\zeta_2|-|\zeta_2-\zeta_1|\ge|z-\zeta_2|-5d_l\ge|z-\zeta_2|-\frac{15}{2}(R_{p+1}-R_p). \tag{2.14}$$

On the other hand, by the choice of q (see (2.2)) and (2.13)

$$|z-\zeta_2|\ge R_m-R_{p+1}\ge R_{p+13}-R_{p+1}-3(R_{p+1}-R_p)=$$

$$=\sum_{s=1}^{12}(R_{p+s+1}-R_{p+s})-3(R_{p+1}-R_p)=\Big(\sum_{s=1}^{12}q^s-3\Big)(R_{p+1}-R_p)>8(R_{p+1}-R_p).$$

The latter inequality and (2.14) yield

$$|z-\zeta_1|\ge|z-\zeta_2|-\frac{15}{2}(R_{p+1}-R_p)>|z-\zeta_2|-\frac{15}{16}|z-\zeta_2|=\frac{1}{16}|z-\zeta_2|.$$

For $l \in \mathcal{L}^-$, $Q^{(l)} \subset \{R_{m+13} \le |\zeta| < 1\}$ we have $p \ge m+13$, and inequality (2.14) still holds.

Similarly, by the choice of q and (2.13)

$$|z-\zeta_2| \ge R_{p-3} - R_{m+1} \ge R_{p-3} - R_{p-12} \ge 9q^4(R_{p+1}-R_p) > 8(R_{p+1}-R_p),$$

that together with (2.14) implies required inequality in this case. Lemma 2 is proved. □

Let $l \in \mathcal{L}^- \cup \mathcal{L}^+$. For $\zeta \in Q^{(l)}$, we define $L(\zeta) = L_l(\zeta) = \log\left(\frac{z-\zeta}{1-\bar{z}\zeta}\right)$, where $\log w$ is an arbitrary branch of $\operatorname{Log} w$ in $w(Q^{(l)})$, $w(\zeta) = \frac{z-\zeta}{1-\bar{z}\zeta}$. Then $L(\zeta)$ is analytic in $Q^{(l)}$. We shall use the following identities

$$L(\zeta) - L(\zeta_l^{(1)}) = \int_{\zeta_l^{(1)}}^{\zeta} L'(s)\,ds = L'(\zeta_l^{(1)})(\zeta-\zeta_l^{(1)}) + \int_{\zeta_l^{(1)}}^{\zeta} L''(s)(\zeta-s)\,ds =$$

$$= L'(\zeta_l^{(1)})(\zeta-\zeta_l^{(1)}) + \frac{1}{2}L''(\zeta_l^{(1)})(\zeta-\zeta_l^{(1)})^2 + \frac{1}{2}\int_{\zeta_l^{(1)}}^{\zeta} L'''(s)(\zeta-s)^2\,ds. \tag{2.15}$$

Elementary geometric arguments show that $|\frac{1}{\bar{z}} - \zeta|^{-1} \le |z-\zeta|^{-1}$ for $z, \zeta \in \mathbb{D}$. Since $L'(\zeta) = \frac{1}{\zeta - z} + \frac{\bar{z}}{1-\bar{z}\zeta}$, we have

$$|L'(\zeta)| \le \frac{2}{|\zeta-z|}, \quad |L''(\zeta)| \le \frac{2}{|\zeta-z|^2}, \quad |L'''(\zeta)| \le \frac{4}{|\zeta-z|^3}. \tag{2.16}$$

Now we estimate $|\Delta_l(z)|$ for $l \in \mathcal{L}^+ \cup \mathcal{L}^-$. By the definitions of $L(\zeta)$, $\Delta_l(z)$, (2.15) and (2.11) we have

$$|\Delta_l(z)| = \Big|\operatorname{Re}\int_{Q^{(l)}} \Big(L(\zeta) - L(\zeta_l^{(1)}) - \frac{1}{2}(L(\zeta_l^{(2)}) - L(\zeta_l^{(1)})\Big)\,d\mu^{(l)}(\zeta)\Big| =$$

$$= \Big|\operatorname{Re}\int_{Q^{(l)}} \Big(L'(\zeta_l^{(1)})\Big(\zeta - \frac{1}{2}(\zeta_l^{(1)} + \zeta_l^{(2)})\Big) +$$

$$+ \int_{\zeta_l^{(1)}}^{\zeta} L''(s)(\zeta-s)ds - \frac{1}{2}\int_{\zeta_l^{(1)}}^{\zeta_l^{(2)}} L''(s)(\zeta_l^{(2)}-s)ds\Big)\,d\mu^{(l)}(\zeta)\Big| =$$

$$= \Big|\operatorname{Re}\int_{Q^{(l)}} \Big(\int_{\zeta_l^{(1)}}^{\zeta} L''(s)(\zeta-s)ds - \frac{1}{2}\int_{\zeta_l^{(1)}}^{\zeta_l^{(2)}} L''(s)(\zeta_l^{(2)}-s)ds\Big)\,d\mu^{(l)}(\zeta)\Big| \tag{2.17}$$

Using (2.17), (2.16) and (2.12), we obtain

$$|\Delta_l(z)| \le \int\limits_{Q^{(l)}} \int\limits_{\zeta_l^{(1)}}^{\zeta} \frac{2|\zeta - s|}{|s-z|^2}|ds|\, d\mu^{(l)}(\zeta) + \frac{1}{2}\int\limits_{Q^{(l)}} \int\limits_{\zeta_l^{(1)}}^{\zeta_l^{(2)}} \frac{2|\zeta_l^{(2)} - s||ds|}{|s-z|^2}\, d\mu^{(l)}(\zeta) \le$$
$$\le 12 d_l^2 \max_{s\in B_l} \frac{1}{|s-z|^2}, \tag{2.18}$$

where $B_l = \overline{U(Q^{(l)}, 2d_l)}$. Applying Lemma 2, we have ($z \in \bar{A}_m$)

$$\sum_{l\in\mathcal{L}^-} |\Delta_l(z)| \le 12 \sum_{l\in\mathcal{L}^-} d_l^2 \max_{s\in B_l} \frac{1}{|s-z|^2} \le C_{11} \sum_{l\in\mathcal{L}^-} \int_{Q^{(l)}} \frac{dm(z)}{|z-\zeta|^2} \le$$
$$\le 4C_{11} \int_{R_{m+13}\le|\zeta|<1} \frac{dm(z)}{|z-\zeta|^2} \le C_{12} \int_{R_{m+13}}^{1} \frac{d\rho}{\rho - |z|} \le C_{13}(q). \tag{2.19}$$

Similarly,

$$\sum_{l\in\mathcal{L}^+} |\Delta_l(z)| \le 12 \sum_{l\in\mathcal{L}^+} d_l^2 \max_{s\in B_l} \frac{1}{|s-z|^2} \le 4C_{11} \sum_{l\in\mathcal{L}^+} \int_{|\zeta|\le R_{m-13}} \frac{dm(z)}{|z-\zeta|^2} \le$$
$$\le C_{12} \int_0^{R_{m-13}} \frac{d\rho}{|z| - \rho} \le C_{14}(q) \log \frac{1}{1-|z|}. \tag{2.20}$$

Hence,

$$\int_{|z|\le R_n} \sum_{l\in\mathcal{L}^+\cup\mathcal{L}^-} |\Delta_l(z)|\, dm(z) < C_{15}(q). \tag{2.21}$$

It remains to estimate $\int_{|z|\le R_n} \sum_{l\in\mathcal{L}^0} |\Delta_l(z)|\, dm(z)$. Here we follow the arguments from [3, e.-g.]. If dist $(z, Q^{(l)}) > 10 d_l$, similarly to (2.17), from (2.15), (2.11),

(2.16) and (2.12) we deduce

$$|\Delta_l(z)| = \Big|\operatorname{Re}\int\limits_{Q^{(l)}}\Big(L'(\zeta_l^{(1)})\Big(\zeta-\frac{1}{2}(\zeta_l^{(1)}+\zeta_l^{(2)})\Big)+$$

$$+\frac{L''(\zeta_l^{(1)})}{2}\Big(\zeta^2-\frac{(\zeta_l^{(1)})^2+(\zeta_l^{(2)})^2}{2}+\zeta_l^{(1)}(\zeta_l^{(1)}+\zeta_l^{(2)}-2\zeta)\Big)+$$

$$+\frac{1}{2}\int_{\zeta_l^{(1)}}^{\zeta}L'''(s)(\zeta-s)^2ds-\frac{1}{4}\int_{\zeta_l^{(1)}}^{\zeta_l^{(2)}}L'''(s)(\zeta_l^{(2)}-s)^2ds\Big)\,d\mu^{(l)}(\zeta)\Big| =$$

$$= \Big|\operatorname{Re}\int\limits_{Q^{(l)}}\Big(\frac{1}{2}\int\limits_{\zeta_l^{(1)}}^{\zeta}L'''(s)(\zeta-s)^2\,ds-\frac{1}{4}\int\limits_{\zeta_l^{(1)}}^{\zeta_l^{(2)}}L'''(s)(\zeta-s)^2\,ds\Big)\,d\mu^{(l)}(\zeta)\Big|\leq$$

$$\leq 6d_l^3\max_{s\in B_l}\frac{1}{|s-z|^3}\leq\frac{6d_l^3}{|\zeta_l^{(1)}-z|^3}\max_{s\in B_l}\Big(1+\frac{|\zeta_l^{(1)}-s|}{|s-z|}\Big)^3\leq\frac{26d_l^3}{|\zeta_l^{(1)}-z|^3}. \qquad (2.22)$$

Since $\mathcal{L}_0$ depends only on m when $z\in A_m$, we have

$$\int\limits_{\overline{A}_m}\sum_{l\in\mathcal{L}^0}|\Delta_l(z)|\,dm(z)\leq\sum_{l\in\mathcal{L}^0}\Big(\int\limits_{\overline{A}_m\setminus U(\zeta_l^{(1)},10d_l)}+\int\limits_{U(\zeta_l^{(1)},10d_l)}\Big)|\Delta_l(z)|\,dm(z)\leq$$

$$\leq\sum_{l\in\mathcal{L}^0}\Big(\int\limits_{\overline{A}_m\setminus U(\zeta_l^{(1)},10d_l)}\frac{26d_l^3}{|z-\zeta_l^{(1)}|^3}dm(z)+\int\limits_{U(\zeta_l^{(1)},10d_l)}|\Delta_l(z)|\,dm(z)\Big). \qquad (2.23)$$

For the first sum we obtain

$$\sum_{l\in\mathcal{L}^0}26d_l^3\int_{\overline{A}_m\setminus U(\zeta_l^{(1)},10d_l)}\frac{1}{|z-\zeta_l^{(1)}|^3}dm(z)\leq\sum_{l\in\mathcal{L}^0}52\pi d_l^3\int_{10d_l}^{2}\frac{tdt}{t^3}\leq$$

$$\leq 6\pi\sum_{l\in\mathcal{L}^0}d_l^2\leq C_{16}\sum_{l\in\mathcal{L}^0}m(Q^{(l)}). \qquad (2.24)$$

We now estimate the second sum. By the definition of $\Delta_l(z)$

$$\Delta_l(z)=\int_{Q^{(l)}}\Big(\log\Big|\frac{z-\zeta}{10d_l}\Big|-\frac{1}{2}\log\Big|\frac{z-\zeta_l^{(1)}}{10d_l}\Big|-\frac{1}{2}\log\Big|\frac{z-\zeta_l^{(2)}}{10d_l}\Big|\Big)\,d\mu^{(l)}(\zeta)-$$

$$-\int_{Q^{(l)}}\Big(\log|1-z\overline{\zeta}|-\frac{1}{2}\log|1-z\overline{\zeta_l^{(1)}}|-\frac{1}{2}\log|1-z\overline{\zeta_l^{(2)}}|\Big)\,d\mu^{(l)}(\zeta)\equiv I_1+I_2.$$

The integral $\int|I_1|\,dm(z)$ is estimated in [3, g.], [4, p.232]. We have

$$\int_{U(\zeta_l^{(1)},10d_l)}|I_1|\,dm(z)\leq C_{17}m(Q^{(l)}). \qquad (2.25)$$

To estimate $|I_2|$ we note that for l sufficiently large, $|z-\zeta| \le 15d_l$, $\zeta \in U(Q^{(l)}, 2d_l)$, $z\in\mathbb{D}$ we have $|\arg z - \arg\zeta| \le 16d_l \le 16(1-|z|)|$ by the choice of q. Hence,

$$|\frac{1}{z}-\bar\zeta| \le \frac{1}{|z|}-1+1-|\zeta|+|\zeta||1-e^{i(\arg\zeta-\arg z)}| \le C'_{17}(1-|z|).$$

Thus, $|1/z-\bar\zeta| \asymp 1-|z|$. Therefore

$$|I_2| \le \int_{Q^{(l)}} \frac{1}{2}\Big|\log\frac{|\frac{1}{z}-\bar\zeta|^2}{|\frac{1}{z}-\zeta_l^{(1)}||\frac{1}{z}-\zeta_l^{(2)}|}\Big| d\mu^{(l)}(\zeta) \le C_{18}.$$

Thus,

$$\int_{U(\zeta_l^{(1)},10d_l)} |I_2|\, dm(z) \le C_{19}(q) m(Q^{(l)}). \tag{2.26}$$

Finnaly, using (2.24)–(2.26) we deduce

$$\int_{\bar A_m} \sum_{l\in\mathcal{L}^0} |\Delta_l(z)|\, dm(z) \le$$

$$\le C_{20}\sum_{l\in\mathcal{L}^0} m(Q^{(l)}) \le 4\pi C_{20}(R^2_{m+13}-R^2_{m-13}) \le C_{21}(q)(R_{m+1}-R_m).$$

Hence, $\int_{|z|\le R_n}\sum_{l\in\mathcal{L}^0}|\Delta_l(z)|dm(z) \le C_{20}(q)$, and this with (2.21) yields that

$$\int_{|z|\le R_n} |V(z)|dm(z) \le C_{22}(q), n\to+\infty. \tag{2.27}$$

Now we construct the function f_1 approximating u_1.

Let $K_n(z) = u_1(z) - \sum_{Q^{(l)}\subset\overline{U(0,R_n)}}\Delta_l(z)$, $K(z)=u_1(z)-V(z)$. By the definition of $\Delta_l(z)$, $K_n\in \mathrm{SH}(\mathbb{D})$ and

$$\mu_{K_n}\Big|_{U(0,R_n)}(z) = \sum_{l=1}^{n}\big(\delta(z-\zeta_l^{(1)})+\delta(z-\zeta_l^{(2)})\big)$$

where $\delta(\zeta)$ is the unit mass supported at $u=0$. For $|z|\le R_n$, $j\ge N\ge n+14$ as in (2.19) we have

$$|K_j(z)-K(z)| \le \sum_{Q^{(l)}\subset\{|\zeta|\ge R_{N+1}\}} |\Delta_l(z)| \le$$

$$\le C_{23}\int_{R_{N+1}\le|\zeta|<1}\frac{dm(z)}{|z-\zeta|^2} \le C_{24}\frac{1-R_{N+1}}{R_{N+1}-|z|} \to 0, \quad N\to+\infty.$$

Therefore $K_n(z)\rightrightarrows K(z)$ on the compact sets in $\mathbb{D}$ as $n\to+\infty$, and $\mu_K\Big|_{\mathbb{D}} = \sum_l(\delta(z-\zeta_l^{(1)})+\delta(z-\zeta_l^{(2)})$. Hence, $K(z)=\log|f_1(z)|$, where f_1 is analytic in $\mathbb{D}$.

2.3. Approximation of u_3

Let u_3 be defined by (2.1),

$$N = 2\big[n(1/2, u_3)/2\big], \quad \rho_0 = \inf\{r \geq 0 : n(r, u_3) \geq N\}.$$

We represent $\mu_{u_3} = \mu^1 + \mu^2$ where μ^1 and μ^2 are measures such that

$$\operatorname{supp}\mu^1 \subset \overline{U(0,\rho_0)}, \quad \operatorname{supp}\mu^2 \subset \overline{U\Big(0,\frac{1}{2}\Big)} \setminus U(0,\rho_0),$$
$$\mu^1\Big(U\Big(0,\frac{1}{2}\Big)\Big) = N, \quad 0 \leq \mu^2\Big(U\Big(0,\frac{1}{2}\Big)\Big) < 2.$$

Let $v_2(z) = \int\limits_{U(0,\frac{1}{2})} \log|z-\zeta|\, d\mu^2(\zeta)$. Then, using the last estimate,

$$\int\limits_{\mathbb{D}} |v_2(z)|\, dm(z) \leq \int\limits_{U(0,1/2)} \int\limits_{\mathbb{D}} |\log|z-\zeta||\, dm(z)\, d\mu^2(\zeta) \leq$$
$$\leq \int\limits_{U(0,1/2)} \int\limits_{U(\zeta,2)} |\log|z-\zeta||\, dm(z)\, d\mu^2(\zeta) \leq C_{25} n\Big(\frac{1}{2}, v_2\Big) \leq 2C_{25}.$$

If $N = 0$ there remains nothing to prove. Otherwise, we have to approximate

$$v_1(z) = u_3(z) - v_2(z) = \int\limits_{\overline{U(0,\rho_0)}} \log|z-\zeta|\, d\mu^1(\zeta). \tag{2.28}$$

In this connection we recall the question of Sodin (Question 2 in [9, p.315]).

Given a Borel measure μ we define the logarithmic potential of μ by the equality

$$\mathcal{U}_\mu(z) = \int \log|z-\zeta|\, d\mu(\zeta).$$

Question. *Let μ be a probability measure supported by the square $\mathcal{Q} = \{z = x+iy : |x| \leq \frac{1}{2}, |y| \leq \frac{1}{2}\}$. Is it possible to find a sequence of polynomials $\mathcal{P}_n$, $\deg \mathcal{P}_n = n$, such that*

$$\iint\limits_{\substack{|x|\leq 1\\ |y|\leq 1}} |n\mathcal{U}_\mu(z) - \log|\mathcal{P}_n(z)||\, dxdy = O(1)\ (n \to +\infty)?$$

We should say that the solution is given essentially in [3], but not asserted. Hence we prove the following

Proposition. *Let μ be a measure supported by the square $\mathcal{Q}$, and $\mu(\mathcal{Q}) = N \in \mathbb{N}$. Then there is an absolute constant C and a polynomial P_N such that*

$$\iint\limits_{\Xi} |\mathcal{U}_\mu(z) - \log|\mathcal{P}_N(z)|| \, dxdy < C,$$

where $\Xi = \{z = x + iy : |x| \leq 1, |y| \leq 1\}$.

Proof of the proposition. As in the proof of Theorem 1, if there are points $p \in \mathcal{Q}$ such that $\mu(\{p\}) \geq 1$ we represent $\mu = \nu + \tilde{\nu}$ where for any $p \in \mathcal{Q}$ we have $\nu(\{p\}) < 1$, and $\tilde{\nu}$ is a finite (at most N summand) sum of the Dirac measures. Then $\mathcal{U}_{\tilde{\nu}} = \log \prod_k |z - p_k|$, so it remains to approximate $\mathcal{U}_\nu$. By Lemma 2.4 [8] there exists a rotation to the system of orthogonal coordinates such that if L is any line parallel to either of the coordinate axes, there is at most one point $p \in L$ with $\nu(\{p\}) > 0$, while always $\nu(L \setminus \{p\}) = 0$. After the rotation the support of the new measure, which is still denoted by ν, is contained in $\sqrt{2}\mathcal{Q}$.

If ω is a probability measure supported on $\mathcal{Q}$, then $\iint_\Xi |\mathcal{U}_\omega(z)| dm(z)$ is uniformly bounded. Therefore we can assume that $N \in 2\mathbb{N}$.

By Theorem E there exists a system (P_l, ν_l) of rectangles and measures $1 \leq l \leq M_\nu$ with the properties: 1) $\nu_l(P_l) = 2$; 2) $\operatorname{supp} \nu_l \subset P_l$; 3) $\sum_l \nu_l = \nu$; 4) every point $s \in \mathcal{Q}$ belongs to interiors of at most four rectangles P_l; 5) ratio of side lengths lays between $1/3$ and 3.

Let

$$\xi_l = \frac{1}{2} \int\limits_{P_l} \xi d\nu_l(\xi). \tag{2.29}$$

be the center of mass of P_l, $1 \leq l \leq M_\nu$.

We define $\xi_l^{(1)}$, $\xi_l^{(2)}$ as solutions of the system

$$\begin{cases} \xi_l^{(1)} + \xi_l^{(2)} = \displaystyle\int\limits_{P_l} \xi d\nu_l(\xi), \\ (\xi_l^{(1)})^2 + (\xi_l^{(2)})^2 = \displaystyle\int\limits_{P_l} \xi^2 d\nu_l(\xi), \end{cases}$$

We have

$$|\xi_l^{(j)} - \xi_l| \leq \operatorname{diam} P_l \equiv D_l, \quad j \in \{1, 2\},$$

$$\max_{\xi \in P_l} |\xi - \xi_l^{(j)}| \leq 2D_l, \ j \in \{1, 2\}, \quad \sup_{\xi \in P_l} |\xi - \xi_l| \leq D_l. \tag{2.30}$$

We write

$$\Omega(z)=\sum_l \int\limits_{P_l}\Bigl(\log\Bigl|z-\xi\Bigr|-\frac{1}{2}\log\Bigl|z-\xi_l^{(1)}\Bigr|-\frac{1}{2}\log\Bigl|z-\xi_l^{(2)}\Bigr|\Bigr)\,d\nu_l(\xi)\equiv$$

$$\equiv\sum_l \delta_l(z). \tag{2.31}$$

Since we have rotated the system of coordinate, it is sufficient to prove that $\int_{\overline{U(0,\sqrt{2})}}|\Omega(z)|\,dm(z)$ is bounded by an absolute constant.

For $\xi\in P_l$, $z\notin P_l$ we define $\lambda(\xi)=\lambda_l(\xi)=\log\bigl(z-\xi\bigr)$, where $\log(z-\xi)$ is an arbitrary branch of $\mathrm{Log}(z-\xi)$ in $z-P_l$. Then $\lambda(\xi)$ is analytic in P_l.

We have

$$|\lambda'''(\xi)|\le\frac{2}{|\xi-z|^3}. \tag{2.32}$$

As in subsection 2.2 we have

$$|\delta_l(z)|\le\Bigl|\operatorname{Re}\int\limits_{P_l}\Bigl(\lambda(\xi)-\lambda(\xi_l^{(1)})-\frac{1}{2}(\lambda(\xi_l^{(2)})-\lambda(\xi_l^{(1)})\Bigr)\,d\nu_l(\xi)\Bigr|\le$$

$$\le\Bigl|\operatorname{Re}\int\limits_{P_l}\Bigl(\frac{1}{2}\int\limits_{\xi_l^{(1)}}^{\xi}\lambda'''(s)(\xi-s)^2\,ds-\frac{1}{4}\int\limits_{\xi_l^{(1)}}^{\xi_l^{(2)}}\lambda'''(s)(\xi-s)^2\,ds\Bigr)\,d\nu_l(\xi)\Bigr|. \tag{2.33}$$

If $\mathrm{dist}\,(z,P_l)>10D_l$ the last estimate and (2.32) yield

$$|\delta_l(z)|\le 24D_l^3\max_{s\in E_l}\frac{1}{|s-z|^3}\le\frac{24D_l^3}{|\xi_l^{(1)}-z|^3}\max_{s\in E_l}\Bigl(1+\frac{|\xi_l^{(1)}-s|}{|s-z|}\Bigr)\le\frac{103D_l^3}{|\xi_l^{(1)}-z|^3},$$

where $E_l=\overline{U(P_l,2D_l)}$.

Then

$$\int\limits_{\overline{U(0,\sqrt{2})}\setminus U(\xi_l^{(1)},10D_l)}\frac{103D_l^3}{|z-\xi_l^{(1)}|^3}dm(z)\le 206\pi D_l^3\int\limits_{10D_l}^{2}\frac{tdt}{t^3}\le$$

$$\le 21\pi D_l^2\le C_{26}m(P_l).$$

On the other hand, by the definition of $\delta_l(z)$

$$\int\limits_{U(\xi_l^{(1)},10D_l)}\delta_l(z)dm(z)=\int\limits_{U(\xi_l^{(1)},10D_l)}\int_{P_l}\Bigl(\log\Bigl|\frac{z-\xi}{10D_l}\Bigr|-$$

$$-\frac{1}{2}\log\Bigl|\frac{z-\xi_l^{(1)}}{10D_l}\Bigr|-\frac{1}{2}\log\Bigl|\frac{z-\xi_l^{(2)}}{10D_l}\Bigr|\Bigr)\,d\nu_l(\xi)dm(z)\le C_{27}m(P_l).$$

From (2.31) and the latter estimates, it follows that

$$\int\limits_{U(0,\sqrt{2})} |\Omega(z)|dm(z) \le \sum_l \int\limits_{U(0,\sqrt{2})} \delta_l(z)dm(z) \le C_{28} \sum_l m(P_l) \le 4C_{28} m(\sqrt{2}\mathcal{Q}) = C_{29}. \tag{2.34}$$

Thus, $\mathcal{P}(z) = \prod_l (z-\xi_l^{(1)})(z-\xi_l^{(2)})$ is a required polynomial. This completes the proof of the proposition. □

Finally, let $f = f_1\mathcal{P}$. By Lemma 1, (2.27), and (2.34) we have ($n \to +\infty$)

$$\int_{|z|\le R_n} |\log|f(z)| - u(z)||\, dm(z) \le \int_{|z|\le R_n} (|K(z) - u_1(z)|| + |u_2(z)| +$$

$$+|\log|\mathcal{P}| - u_3(z)|)\, dm(z) \le \int_{|z|\le R_n} (|V(z)| + |\Omega(z)|)\, dm(z) + C_{10}(q) \le C_{30}(q).$$

Fixing any q satisfying (2.2) we finish the proof of Theorem 1.

3 Uniform approximation

In this section we prove some counterparts of results due to Yu.Lyubarskii and Eu.Malinnikova [3]. We start with counterparts of notions introduced in [3], which reflect regularity properties of measures.

Definition 1. Let $b\colon [0,1) \to (0,+\infty)$ be such that $b(r) \le 1-r$,

$$b(r_1) \asymp b(r_2) \quad as \quad 1-r_1 \asymp 1-r_2, \quad r_1 \uparrow 1. \tag{3.1}$$

A measure μ on $\mathbb{D}$ admits a *partition of slow variation with the function* b if there exist integers N, p and sequences $(Q^{(l)})$ of subsets of $\mathbb{D}$ and $(\mu^{(l)})$ of measures with the following properties:

i) $\operatorname{supp}\mu^{(l)} \subset Q^{(l)}$, $\mu^{(l)}(Q^{(l)}) = p$;

ii) $\operatorname{supp}(\mu - \sum_l \mu^{(l)}) \subset \mathbb{D}$, $(\mu - \sum_l \mu^{(l)})(\mathbb{D}) < +\infty$.

iii) $1 - \operatorname{dist}(0, Q^{(l)}) \ge K(p)\operatorname{diam} Q^{(l)}$, and each $z \in \mathbb{D}$ belongs to at most N various $Q^{(l)}$'s;

iv) For each l the set $\log Q^{(l)}$ is a rectangle with sides parallel to the coordinate axes, and the ratio of sides lengths lies between two positive constants independent of l.

v) $\operatorname{diam} Q^{(l)} \asymp b(\operatorname{dist}(Q^{(l)}, 0))$.

Remark 3.1. This is similar to [3], except we have introduced the parameter p ($p = 2$ in [3]). Property iii) is adapted for $\mathbb{D}$.

Definition 2. Given a function b satisfying (3.1) we say that a *measure* μ *is locally regular with respect to (w.r.t.)* b *if*

$$\int_0^{b(|z|)} \frac{\mu(U(z,t))}{t} dt = O(1), \quad r_0 < |z| < 1,$$

for some constant $r_0 \in (0,1)$.

Theorem 3. *Let* $u \in \mathrm{SH}(D)$*,* $b\colon [0,1) \to (0,+\infty)$ *satisfy (3.1). Let* μ_u *admits a partition of slow variation, assume that* μ_u *is locally regular w.r.t.* b*, and, with* p *from above, that*

$$\int_0^1 \frac{b^{p-1}(t)}{(1-t)^p} dt < +\infty. \tag{3.2}$$

Then there exists an analytic function f *in* $\mathbb{D}$ *such that* $\forall \varepsilon > 0 \ \exists r_1 \in (0,1)$

$$\log|f(z)| - u(z) = O(1), \quad r_1 < |z| < 1, \ z \notin E_\varepsilon$$

where $E_\varepsilon = \{z \in \mathbb{D} : \mathit{dist}(z, Z_f) \le \varepsilon b(|z|)\}$*, and for some constant* $C > 0$

$$\log|f(z)| - u(z) < C, \quad z \in \mathbb{D}. \tag{3.3}$$

Moreover, $Z_f \subset U(\mathit{supp}\,\mu_u, K_1(p)b(|z|))$*,* $K_1(p)$ *is a positive constant, and*

$$T(r,u) - T(r, \log|f|) = O(1), \quad r \uparrow 1. \tag{3.4}$$

Remark 3.2. The author does not know whether condition (3.2) is necessary. But if $b(t) = O((1-t)\log^{-\eta}(1-t)$, $\eta > 0$, $t \uparrow 1$ (3.2) holds for sufficiently large p. On the other hand, in view of v) the condition $b(t) = O(1-t)$ as $t \uparrow 1$ is natural.

Proof of Theorem 3. We follow [3] and also use arguments and notation from the proof of Theorem 1.

Let $\tilde{\mu} = \mu_u - \sum_l \mu^{(l)}$. Since $\left|\frac{z-\zeta}{1-\bar{\zeta}z}\right| \to 1$ as $|z| \uparrow 1$ for fixed $\zeta \in \mathbb{D}$, $\tilde{\mu}(\mathbb{D}) < +\infty$,

$$\tilde{u}_1(z) = \int_{\mathbb{D}} \log\left|\frac{z-\zeta}{1-\bar{\zeta}z}\right| d\tilde{\mu}(\zeta)$$

is a subharmonic function in $\mathbb{D}$ and $|\tilde{u}_1(z)| < C$ for $r_1 < |z| < 1$, $r_1 \in (0,1)$. So we can assume that $\mu_u = \sum_l \mu^{(l)}$ where $\mu^{(l)}$ are from Definition 1.

Fix a partition of slow variation. Instead of points $\zeta_l^{(1)}$ and $\zeta_l^{(2)}$ satisfying (2.11) we define $\xi_1^{(l)}, \ldots, \xi_p^{(l)}$ from the system

$$\begin{cases} \xi_1 + \cdots + \xi_p & = \int_{Q^{(l)}} \xi d\mu^{(l)}(\xi), \\ \xi_1^2 + \cdots + \xi_p^2 & = \int_{Q^{(l)}} \xi^2 d\mu^{(l)}(\xi), \\ & \vdots \\ \xi_1^p + \cdots + \xi_p^p & = \int_{Q^{(l)}} \xi^p d\mu^{(l)}(\xi), \end{cases} \tag{3.5}$$

Lemma 3 is a modification of the estimates in(2.12).

Lemma 3. *Let* Π *be a set in* $\mathbb{C}$, μ *is a measure on* Π, $\mu(\Pi) = p \in \mathbb{N}$, $\operatorname{diam} \Pi = d$. *Then for any solution* $(\xi_1, \ldots, \xi_p)$ *of (3.5) we have* $|\xi_j - \xi_0| \le K_1(p) d$ *where* $K_1(p)$ *is a constant,* ξ_0 *is the center of mass of* Π.

Proof of Lemma 3. Let $\xi_0 = \frac{1}{p} \int_\Pi \xi d\mu(\xi)$ be the center of mass of Π. By induction, it is easy to prove that (3.5) is equivalent to the system

$$\begin{cases} w_1 + & \cdots + w_p = 0, \\ w_1^2 + & \cdots + w_p^2 = J_2, \\ & \vdots \\ w_1^p + & \cdots + w_p^p = J_p, \end{cases} \tag{3.6}$$

where $w_k = \xi_k - \xi_0$, $J_k = \int_\Pi (\xi - \xi_0)^k \, d\mu(\xi)$, $1 \le k \le p$. Note that

$$|J_k| \le \int_\Pi |\xi - \xi_0|^k \, d\mu(\xi) \le p d^k.$$

From algebra it is well-known that the symmetric polynomials

$$\sum_{1 \le i_1 < \cdots < i_k \le m} w_{i_1} \cdots w_{i_k},$$

$1 \le k \le m$, can be obtained from the polynomials $\sum_{j=1}^m w_j^k$ using only finite number of operations of addition and multiplication. Therefore (3.6) yields

$$\begin{cases} w_1 + \cdots + w_p = 0, \\ \sum_{1 \le i_1 < i_2 \le p} w_{i_1} w_{i_2} = b_2, \\ \quad \vdots \\ w_1 \cdots w_p = b_p, \end{cases}$$

where $b_k = \sum_l a_{lk}(J_1)^{s_{1l}^{(k)}} \cdots (J_m)^{s_{ml}^{(k)}}$, $a_{lk} = a_{lk}(p)$, $s_{jl}^{(k)}$ are non-negative integers, and $\sum_{j=1}^{p} s_{jl}^{(k)} j = k$. The last equality follows from homogeneousity. Hence, there exists a constant $K_1(p) \geq 2$ such that $|b_k| \leq K_1(p)d^k$, $1 \leq k \leq p$. By Vieta's formulas ([11, §§51,52]) w_j, $1 \leq j \leq p$, satisfy the equation

$$w^p + b_2 w^{p-2} - b_3 w^{p-3} + \cdots + (-1)^p b_p = 0. \tag{3.7}$$

For $|w| = K_1(p)d$ we have

$$|w^p + b_2 w^{p-2} - b_3 w^{p-3} + \cdots + (-1)^p b_p| \leq K_1(p)(d^2|w|^{p-2} + \cdots + d^p) =$$
$$= K_1(p)d^p(K_1^{p-2} + K_1^{p-1} + \cdots + 1) < 2K_1^{p-1}(p)d^p \leq K_1^p(p)d^p = |w|^p.$$

By Rouché's theorem all p roots of (3.7) lay in the disk $|w| \leq K_1(p)d$, i.e. $|\xi_j - \xi_0| \leq K_1(p)d$. Consequently, $\operatorname{dist}(\xi_j, \Pi) \leq K_1(p)d$. $\square$

Applying Lemma 3 to $Q^{(l)}$ we obtain that $|\xi_l^{(j)} - \xi_l| \leq K_1(p)d_l$, $1 \leq j \leq p$, where $\xi_l = \frac{1}{p}\int_{Q^{(l)}} \xi d\mu^{(l)}(\xi)$.

Consider

$$V(z) = \sum_l j_l(z) \stackrel{\text{def}}{=} \sum_l \int_{Q^{(l)}} \Big(\log\Big|\frac{z-\zeta}{1-\bar{z}\zeta}\Big| - \frac{1}{p}\sum_{j=1}^{p} \log\Big|\frac{z-\xi_l^{(j)}}{1-\bar{z}\xi_l^{(j)}}\Big|\Big) d\mu^{(l)}(\zeta).$$

For $R_n = 1 - 2^{-n}$, $z \in A_m$, m is fixed, we define sets of indices $\mathcal{L}^+$, $\mathcal{L}^-$ and $\mathcal{L}^0$ as in the proof of Theorem 1.

The estimate of $\sum\limits_{l\in\mathcal{L}^-} j_l(z)$ repeats that of $\sum\limits_{l\in\mathcal{L}^-} \Delta_l(z)$, so

$$\sum_{l\in\mathcal{L}^-} |j_l(z)| \leq C_{31}. \tag{3.8}$$

Following [3] we estimate $\sum_{l\in\mathcal{L}^0} j_l(z)$. Let $b_m = b(R_m)$. Note that $d_l \asymp b_m$ for $l \in \mathcal{L}^0$ by condition v). As in (2.18) we have

$$|j_l(z)| \leq C_{32} d_l^3 \max_{s\in U(Q^{(l)}, K_1(p)d_l)} |s-z|^{-3} \leq C'_{32} \frac{d_l^3}{|\xi_l^{(1)} - z|^3}, \tag{3.9}$$

provided that $\operatorname{dist}(z, Q^{(l)}) \geq 3K_1(p)d_l$. Then

$$\Big|\sum_{\substack{l\in\mathcal{L}^0 \\ Q^{(l)}\cap U(z, 3K_1(p)d_l)=\varnothing}} j_l(z)\Big| \leq C_{32} \sum_{l\in\mathcal{L}^0} \frac{d_l^3}{|\xi_l^{(1)} - z|^3} \leq$$
$$\leq C_{33} b_m \int\limits_{|z-\zeta|>C_{34}b_m} \frac{dm(\zeta)}{|z-\zeta|^3} \leq C_{35}\frac{b_m}{b_m} = C_{35}. \tag{3.10}$$

Let now l be such that $Q^{(l)} \cap U(z, 3K_1(p)d_l) \neq \varnothing$. Since $d_l \asymp b_m$, the number of these l is bounded uniformly in l. For $z \notin E_\varepsilon$ we have $(1 \leq k \leq p)$

$$\log|z-\xi_l^{(k)}| = \log b_m + \log\frac{|z-\xi_l^{(k)}|}{b_m} = \log b(|z|) + O(1). \tag{3.11}$$

Therefore

$$j_l(z) = \int_{Q^{(l)}}\Big(\log|z-\zeta| - \frac{1}{p}\sum_{k=1}^{p}\log|z-\xi_l^{(k)}|\Big)d\mu^{(l)}(\zeta)-$$
$$-\frac{1}{p}\int_{Q^{(l)}}\log\frac{|1-\bar z\zeta|^p}{\prod_{k=1}^{p}|1-\bar z\xi_l^{(k)}|}d\mu^{(l)}(\zeta) = J_3 + J_4.$$

As in the proof of the proposition (see the estimate of I_2), one can show that Since $|\frac{1}{\bar z}-\zeta| \asymp 1-|z| \asymp |\frac{1}{\bar z}-\xi_l^{(j)}|$. Hence, we have $J_4 = O(1)$.

Let $\mu_z(t) = \mu(\overline{U(z,t)})$. Further, using (3.11),

$$J_3 = \int\limits_{Q^{(l)}\setminus U(z,b(|z|))}\log|z-\zeta|d\mu^{(l)}(\zeta) + \int\limits_{U(z,b(|z|))}\log|z-\zeta|\,d\mu^{(l)}(\zeta)-$$
$$-p\log b(|z|) + O(1) = \mu^{(l)}(Q^{(l)}\setminus U(z,b(|z|))\log b(|z|) + O(1)+$$
$$+\int_0^{b(|z|)}\log t\,d\mu_z^{(l)}(t) - p\log b(|z|) = \mu^{(l)}(Q^{(l)}\setminus U(z,b(|z|))\log b(|z|)+$$
$$+O(1) + \mu^{(l)}(U(z,b(|z|))\log b(|z|) - \int_0^{b(|z|)}\frac{\mu_z^{(l)}(t)}{t}dt-$$
$$-p\log b(|z|) = -\int_0^{b(|z|)}\frac{\mu_z^{(l)}(t)}{t}dt + O(1) = O(1) \tag{3.12}$$

by the regularity of μ_u w.r.t $b(t)$. Together with (3.10) it yields

$$\sum_{l\in\mathcal{L}^0}|j_l(z)| = O(1), \quad z \notin E_\varepsilon. \tag{3.13}$$

Now we estimate $\sum_{l\in\mathcal{L}^+} j_l(z)$. Integration by parts gives us

$$L(\zeta)-L(\xi_l^{(1)}) = \sum_{k=1}^{m}\frac{1}{k!}L^{(k)}(\xi_l^{(1)})(\zeta-\xi_l^{(1)})^k + \frac{1}{m!}\int_{\xi_l^{(1)}}^{\zeta}L^{(m+1)}(s)(\zeta-s)^m\,ds, \tag{3.14}$$

where $L(\zeta) = \log\frac{z-\zeta}{1-\bar z\zeta}$,

$$|L^{(k)}(\zeta)| \leq \frac{2(k-1)!}{|z-\zeta|^k} \tag{3.15}$$

The definition of $\xi_l^{(k)}$, $1 \le k \le p$ allows us to cancel the first p moments. Therefore, similarly to (2.18) and (2.22) we have

$$|j_l(z)| \le C_{28} d_l^{p+1} \max_{s \in U(Q^{(l)}, K_1(p)d_l)} |s-z|^{-p-1}. \tag{3.16}$$

Then ($z \in A_m$)

$$\sum_{l \in \mathcal{L}^+} |j_l(z)| \le C_{28} \sum_{l \in \mathcal{L}^+} \frac{d_l^{p+1}}{|z-\xi_l^{(1)}|^{p+1}} \le C_{29} \sum_{l \in \mathcal{L}^+} d_l^{p-1} \int_{Q^{(l)}} \frac{dm(z)}{|z-\zeta|^{p+1}} \le$$
$$\le C_{30}(N,p,q) \sum_{n \le m-12} b^{p-1}(R_n) \int_{\bar{A}_n} \frac{dm(z)}{|z-\zeta|^{p+1}} \le C_{31} \int\limits_{|\zeta| \le R_{m-12}} \frac{b^{p-1}(|\zeta|)dm(\zeta)}{|z-\zeta|^{p+1}} \le$$
$$\le C_{32} \int_0^{R_{m-12}} \frac{b^{p-1}(\rho)}{(|z|-\rho)^p} d\rho \le C_{33} \int_0^1 \frac{b(\rho)^{p-1}}{(1-\rho)^p}\, d\rho < +\infty.$$

Using the latter inequality, (3.13) and (3.8) we obtain $|V(z)| = O(1)$ for $z \notin E_\varepsilon$.

The construction of f is similar to that of Theorem 1. It remains to prove (3.3) for $z \in E_\varepsilon$.

By (3.10) it is sufficient to consider l with $Q^{(l)} \cap U(z, 3K_1(p)d_l) \ne \varnothing$. For all sufficiently large $l \in \mathcal{L}^0$ we have

$$\Big| \int_{Q^{(l)}} \log|z-\zeta| d\mu^{(l)}(\zeta) \Big| \le \int\limits_{U(z, 4K_1(p)d_l)} \log \frac{1}{|z-\zeta|} d\mu^{(l)}(\zeta) \le$$
$$\le \int\limits_0^{4K_1(p)d_l} \log\frac{1}{t} d\mu_z^{(l)}(t) = \log \frac{1}{4K_1(p)d_l} \mu_z^{(l)}(4K_1(p)d_l) + \int\limits_0^{4K_1(p)d_l} \frac{\mu_z^{(l)}(t)}{t} dt = O(1). \tag{3.17}$$

Then we have

$$\log|f(z)| - u(z) = O(1) + \sum_{l \in \mathcal{L}^0} \Big(\sum_{k=1}^{p} \log|z-\xi_l^{(k)}| - \int_{Q^{(l)}} \log|z-\zeta| d\mu^{(l)}(\zeta) \Big) < C,$$

because $|z-\xi_l^{(j)}| = O(b(|z|)) < 1$ for $l \ge l_0$ and (3.3) is proved.

Finally, in order to prove (3.4) we note that for $z \in E_\varepsilon$ in view of (3.8), (3.10), (3.17) we have

$$\log|f(z)| - u(z) = \sum_{j=1}^{m} \log|z-\zeta_j| + O(1)$$

where $\zeta_j \in Z_f$, and m are uniformly bounded. Then $T(r, u - \log|f|)$ is bounded, and consequently

$$T(r,u) = T(r, \log|f|) + T(r, u - \log|f|) + O(1) = T(r, \log|f|) + O(1).$$

$\square$

Proof of Theorem 2. Let $\mu_j = \mu_u\Big|_{[\gamma_j]}$. By the assumptions of the theorem we have $\mu_u = \sum_{j=1}^m \mu_j$. We can write $u = \sum_{j=1}^m u_j$, where $u_j \in \mathrm{SH}(\mathbb{D})$, and $\mu_{u_j} = \mu_j$. Therefore, it is sufficient to approximate each u_j, $1 \le j \le m$, separately.

We write $R(r) = (1-r)^{-1}$ and $W(R) = R^{\rho(R)}$. Then $\mu_j(U(0,r)) = \Delta_j W(R(r))$. Put $b(t) = (1-t)/W(R(r))$. Then condition (3.2) is satisfied. We are going to prove that μ_j admits a partition of slow variation and is locally regular w.r.t. $b(t)$. We define a sequence (r_n) from the relation $\Delta_j W(R(r_n)) = 2n$, $n \in \mathbb{N}$. Then using the theorem on the inverse function, and properties of the proximate order [7, Ch.1, §12] we have $(r' \in (r_n, r_{n+1}))$

$$r_{n+1} - r_n = \frac{2(1-r')^2}{\Delta_j W'(R(r'))} = \frac{(2+o(1))R(r')(1-r)'^2}{\Delta_j \sigma W(R(r'))} = \frac{2+o(1)}{\Delta_j \sigma} b(r') \asymp b(r_n).$$

Let $Q^{(n)} = \{z : r_n \le |z| \le r_{n+1}, \varphi_n^- \le \theta \le \varphi_n^+\}$ where $\varphi_n^- = \theta_j(r_n) - K(r_{n+1} - r_n)$, $\varphi_n^+ = \theta_j(r_n) + K(r_{n+1} - r_n)$. Since $|\theta_j'(t)| \le K$, we have $\theta_j(r) \in [\varphi_n^-, \varphi_n^+]$, $r_n \le r \le r_{n+1}$. Let $\mu^{(n)} = \mu_j\Big|_{Q^{(n)}}$. Then, by the definition of r_n, $\mu^{(n)}(Q^{(n)}) = 2$. Therefore conditions i) and iv) in the definition of a partition of slow growth are satisfied. Condition ii) is trivial. Since $\operatorname{diam} Q^{(n)} \asymp b(r_n) \asymp (1-r_n)^{1+\sigma(r_n)}$, $\sigma > 0$, conditions iii) and v) are valid. Therefore, μ admits a partition of slow growth w.r.t. b, $N = p = 2$.

Finally, we check the local regularity of μ_j w.r.t. $b(t)$. For $|z| = r$, $\rho \le b(r)$ we have

$$\mu_j(U(z,\rho)) \le \Delta_j W(R(r+\rho)) - \Delta_j W(R(r-\rho)) = W'(R(r^*)) \frac{2\rho\Delta_j}{(1-r^*)^2} =$$
$$= (2+o(1))\Delta_j \sigma \rho \frac{W(R(r^*))}{1-r^*} \le \frac{3\sigma\rho\Delta_j}{b(r)}.$$

Then $\int_0^{b(r)} \frac{\mu(U(z,\rho))}{\rho} d\rho \le 3\sigma\Delta_j$ as required.

Applying Theorem 3 we obtain (1.8), (1.9), and (1.10) for some analytic function f_j in $\mathbb{D}$.

Finally, we define $f = \prod_{j=1}^m f_j$.

The theorem is proved. $\square$

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